

# Criterion for the occurrence of many body localization in the presence of a single particle mobility edge

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In one dimension, noninteracting particles can undergo a localization-delocalization transition in a quasiperiodic potential. This transition transforms into a Many-Body Localization (MBL) transition upon the introduction of interactions. Mobility edges can also appear in the single particle spectrum for certain types of quasiperiodic potentials. In a recent study [Phys. Rev. Lett. 115,230401 (2015)], we investigated the effect of interactions in two models with such mobility edges and observed MBL in one and its absence in the other. In this work, we propose a criterion to determine whether MBL is likely to occur in the presence of interactions based on the properties of the non-interacting model. The relevant quantity to calculate is a weighted ratio of the participation ratios of the delocalized and localized states. This quantity provides a measure of how localized the localized states are relative to how delocalized the delocalized states are in the non-interacting model.

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*Introduction:* An arbitrarily weak amount of disorder can localize all eigenstates in a non-interacting quantum system in dimensions  $d \leq 2$  while in three dimensions a mobility edge of energy  $E_c$  can exist that separates localized and delocalized states [1–3]. In the presence of interactions, for large enough disorder, a many-body version of this effect called Many-Body Localization (MBL) can occur [4], which has attracted a lot of attention recently [5–8]. Systems displaying MBL are the only known examples of generic interacting isolated quantum systems that do not thermalize [9, 10] and thus do not obey the Eigenstate Thermalization Hypothesis (ETH) [11–13]. It has been argued these systems possess emergent conservation laws [14–17] which prevent thermalization like in integrable systems [18, 19]. Systems which display MBL typically undergo a thermal-MBL transition as a function of quenched disorder.

A delocalization-localization transition analogous to the thermal-MBL transition can occur even in a non-interacting system. An example of a one-dimensional microscopic model with such a transition is the Aubry-Andre model (AA model) [20]. The transition is between a phase with *all* single particle states localized and one with *all* states delocalized. The delocalization-localization transition in this model transforms into a thermal-MBL transition upon the introduction of interactions [21]. the AA model has recently been emulated in experiments on cold-atoms in the non-interacting limit [22, 23] and with interactions to observe MBL [24].

The AA model can be modified to yield single-particle mobility edges [25–27]. Such mobility edges can also be seen in models with correlated disorder [39]. It was argued by Nandkishore and Potter that a non-interacting system with coexisting localized and *protected* delocalized states will thermalize upon the introduction of weak interactions if  $\nu d \geq 1$  [29].  $d$  here is the number of physical dimensions and the localization length  $\xi$  diverges at an energy  $E_c$  as  $\xi \sim |E - E_c|^{-\nu}$ . A question that arises is

whether MBL could be present in a system with localized and *unprotected* delocalized states upon the introduction of weak interactions. Such a situation is *generic* in disordered three dimensional and systems thus, the above question is of relevance to interacting solid-state systems. Recent works [31, 32, 38] have shown that MBL can indeed occur in models with single particle mobility edges in one dimension. However, it was also seen that not all such systems with unprotected delocalized states display MBL upon the introduction of weak interactions [31, 38]. It is thus natural to ask if there is a criterion that can be formulated by an examination of the single particle spectrum of the non-interacting model to determine whether MBL can develop upon the introduction of weak interactions. The identification of such a criterion would be significant because the required calculations would have to be performed only on non-interacting models (and not interacting ones), which can be done analytically or numerically on fairly large system sizes.

In this work, based on numerical exact diagonalization studies of several different models, we propose a criterion to determine whether a system with a single particle mobility edge displays MBL upon turning on weak interactions. Specifically, we argue that the relevant quantity is  $\epsilon$ , which can be calculated from the non-interacting spectrum of a system of size  $L$  and is defined as

$$\epsilon = \frac{\eta(1 - MPR_D/L)}{(MPR_L - 1)}, \quad (1)$$

$\eta$  is the ratio of the number of localized states to delocalized single particle states,  $MPR_D$  is the mean participation ratio (PR) of the delocalized states and  $MPR_L$  is the mean participation ratio (PR) of the localized states. (The PR of a normalized eigenstate  $\psi$  is defined  $PR_\psi = 1/\sum_j |c_j|^4$ , where  $c_j$  is the amplitude of  $\psi$  at site  $j$ .  $PR \sim 1$  for a localized state and is much larger (typical  $\sim L$ ) for a delocalized one.) This quantity  $\epsilon$  is thus a measure of the relative strengths of the localized and

delocalized states in the non-interacting model, i.e. how strongly localized the localized states are compared to how strongly delocalized the delocalized ones are. Our main result is that the system remains localized (thermalizes) upon the introduction of weak interactions when  $\epsilon > (<) 1$ . This also serves as a criterion to detect the thermal-MBL transition in these systems, based on the properties of the non-interacting system. We also comment on whether the exponent  $\nu$  defined in the previous paragraph is significant in determining whether MBL occurs in systems with mobility edges.

We have studied five different interacting one-dimensional models of spinless fermions, which have single-particle mobility edges in the non-interacting limit ( $V = 0$ ). Three of the models are of the form

$$H = \sum_i h_i n_i - t(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) + V n_i n_{i+1}, \quad (2)$$

where where  $c$  ( $c^\dagger$ ) annihilates (creates) spinless fermions,  $t$  is the hopping, and  $h_i$  is a quasi-periodic potential.  $V$  is the interaction between the fermions on neighboring sites. The models differ in the specific form of  $h_i$  but all of them involve an irrational number  $\alpha$  and offset  $\phi$ .

For the first model (model I),  $h_i = h \frac{\cos(2\pi i\alpha + \phi)}{1 - \beta \cos(2\pi i\alpha + \phi)}$ , with  $\beta \in (-1, 1)$ . When  $\beta = 0$  and  $V = 0$ , it reduces to the AA model. For  $V = 0$  and  $\beta \neq 0$ , there is a mobility edge separating, localized and extended states at an energy  $E$  given by  $\beta E = 2 \operatorname{sgn}(h)(|t| - |h|/2)$  [27].

The second model (model II) is described by  $h_i = h \frac{1 - \cos(2\pi i\alpha + \phi)}{1 + \beta \cos(2\pi i\alpha + \phi)}$ , with  $\beta \in (-1, 1)$ . When  $V = 0$ , this model also reduces to the AA model for  $\beta = 0$  and for  $\beta \neq 0$ , there is a mobility edge separating, localized and extended states at an energy  $E$  given by  $\beta E = 2 \operatorname{sgn}(h)(|t| - |h|/2)$  [27].

The third model (model III) is described by  $h_i = h \cos(2\pi \alpha i^n + \phi)$  with  $0 < n < 1$ . For  $V = 0$  and  $n = 1$ , this is just the AA model. However, for  $n < 1$  and  $V = 0$ , the model has a single-particle mobility edge when  $h < 2t$  [25, 26]. All single particle states with energy between  $\pm|2t - h|$  are delocalized and all other states are localized. For  $h > 2t$  all single particle states are localized as in the usual AA model.

The fourth model (model IV) is also of the form of Eqn. 2 with  $h_i = h \cos(2\pi \alpha i + \phi)$  but with an additional next-nearest neighbor hopping term  $-t'(c_i^\dagger c_{i+2} + c_{i+2}^\dagger c_i)$ . This model has both localized and delocalized states in certain range of parameters for  $V = 0$  [33].

The fifth model (model V) is also of the form of Eqn. 2 but without a quasi-periodic potential. Instead  $h_i = \sum_{k=1}^{L/2} (k^{-\gamma} (2\pi/L)^{(1-\gamma)})^{1/2} \cos(2\pi i k/L + \phi_k)$ .  $\phi_k$  are  $L/2$  independent random phases uniformly distributed in the interval  $[0, 2\pi]$  and we use the normalization  $\sqrt{\langle h_i^2 \rangle - \langle h_i \rangle^2} = 1$ . Unlike the other models, this one has long-range correlated disorder. This

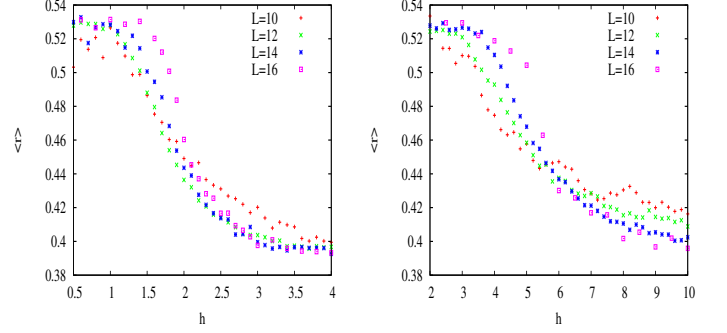


FIG. 1: (Color Online) The variation of the mean of the ratio between adjacent gaps in the spectrum with the strength of incommensurate potential  $h$  for  $L = 10, 12, 14, 16$  at half filling for (left) model I ( $V = 1$  and  $\beta = -0.6$ ) and (right) model II ( $V = 1$  and  $\beta = -0.75$ ). The thermal to MBL transition can be estimated from the crossing of the different curves and is at  $h \approx 2$  for model I and  $h \approx 6$  for model II.

model has both localized and delocalized states for  $\gamma > 2$  and  $V = 0$  [39].

We have studied all five models using exact diagonalization on finite-sized systems up to size  $L = 16$  and have averaged over the offset  $\phi$  for better statistics. We set  $t = 1$  and  $\alpha = \frac{\sqrt{5}-1}{2}$  and all our calculations are done at half filling.

We first determine which of the above models have MBL phases and locate the thermal-MBL transition in them. We do this by studying the energy level-spacing distribution and scaling of the entanglement entropy of mid-spectrum states as explained below.

*Energy level spacing statistics:* Energy level spacing statistics can be used to estimate the location of the MBL transition. At the transition the statistics change from being Wigner-Dyson-like (characteristic of the ergodic phase) to Poissonian (characteristic of the MBL phase) and can be tracked by the ratio of successive gaps,  $r_n = \frac{\min(\delta_n, \delta_{n+1})}{\max(\delta_n, \delta_{n+1})}$  [9], where  $\delta_n = E_{n+1} - E_n$ , the difference in energy between the  $n^{\text{th}}$  and  $n+1^{\text{st}}$  energy eigenvalues. For a Poissonian distribution the mean value of  $r$  is  $2 \ln 2 - 1 \approx 0.386$  while for a Wigner-Dyson-like distribution, specifically of the Gaussian Orthogonal (GOE) type) distribution as is appropriate here, it is  $\approx 0.5295$ . The distribution function  $P(r) \rightarrow 0$ , as  $r \rightarrow 0$  in the presence of level repulsion.

For model I, with  $V = 1$  and  $\beta = -0.6$  and model II, with  $V = 1$  and  $\beta = -0.75$  as  $h$  is increased, the level spacing distribution changes from the GOE distribution to Poissonian distribution for system size  $L = 10, 12, 14, 16$  as shown in Fig. 1. The data for different system sizes cross near  $h \sim 2$  for model I and  $h \sim 6$  for model II. Hence,  $h = 2$  and  $h = 6$  can be con-

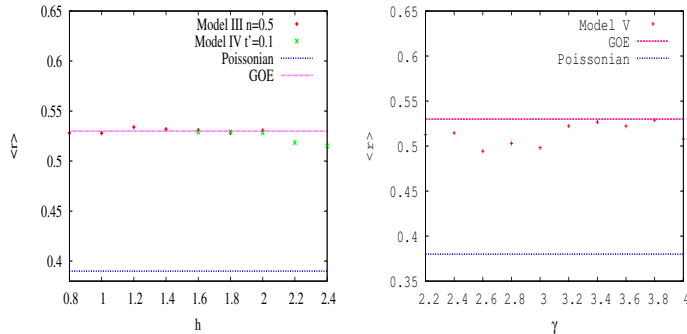


FIG. 2: (Color Online) The variation of the mean of the ratio between adjacent gaps in the spectrum (left) with the strength of incommensurate potential  $h$  for  $L = 16$  at half filling for model III ( $V = 1$  and  $n = 0.5$ ) and model IV ( $V = 1$  and  $t' = 0.1$ ) and (right) with  $\gamma$  for model V. Blue and Pink dashed lines correspond to Poissonian and GOE ensemble. It can be seen that unlike models I and II, models III, IV and V are always in the thermal phase.

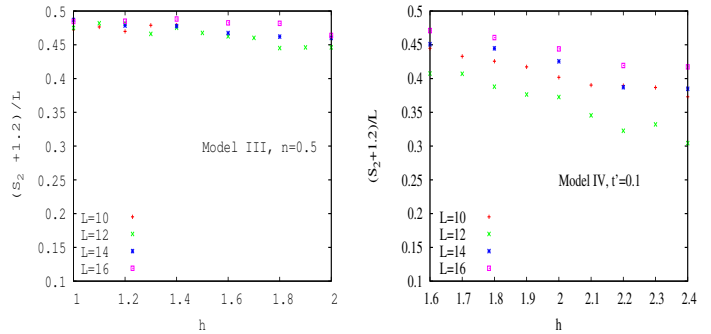


FIG. 4: (Color Online) Variation of  $(S_2 + 1.2)/L$  (where  $S_2$  is the Renyi entropy) of a typical mid-gap state for  $L = 10, 12, 14, 16$  at half filling for (left) model III ( $V = 1$  and  $n = 0.5$ ) and (right) model IV ( $V = 1$  and  $t' = 0.1$ ). The entanglement entropy always appears to be proportional to  $L$  (volume law) which indicates thermalization as opposed to the behavior seen for models I and II. This is consistent with the results obtained from the energy level spacing distribution.

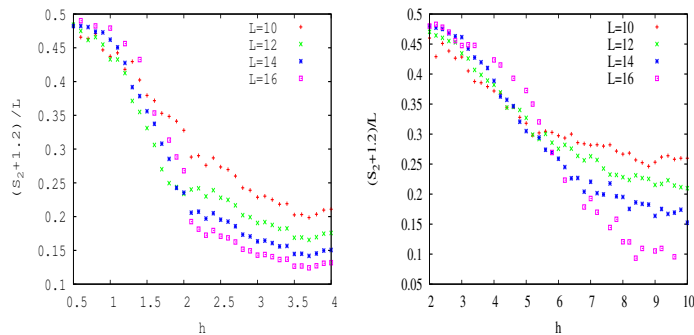


FIG. 3: (Color Online) Variation of  $(S_2 + 1.2)/L$  (where  $S_2$  is the Renyi entropy) of a typical mid-gap state for  $L = 10, 12, 14, 16$  at half filling for (left) model I ( $V = 1$  and  $\beta = -0.6$ ) and (right) model II ( $V = 1$  and  $\beta = -0.75$ ). The thermal-MBL transition can be estimated from the point where the curves for different values of  $L$  appear to separate from one another. This is seen to be at  $h \approx 2$  for model I and  $h \approx 6$  for model II consistent with the values obtained from the energy level spacing distribution.

sidered as the locations of the thermal-MBL transition for models I and II respectively.

On the other hand, model III, model IV and model V do not show a thermal-MBL transition and the level spacing distributions of these models are of the GOE type as shown in Fig. 2. Even though the non-interacting versions of these models have single particle mobility edges, switching on interactions causes them to thermalize.

*Eigenstate entanglement entropy:* The entanglement entropy can also be used to distinguish between the thermal and many-body localized phases of a model. For a typical eigenstate (i.e. one from the middle of the spectrum), it obeys a volume law in the thermal phase and an area law in the MBL phase. We calculate the order 2 Renyi entropy  $S_2 = -\log_2(\text{Tr}_A \rho_A^2)$  between the two halves  $A$  and  $B$  of a system of length  $L$  [34]. This is computationally less expensive to calculate than the von-Neumann entropy and has also recently been measured in experiments [35].  $\rho_A$  is the reduced density matrix of  $A$  obtained by tracing out the degrees of freedom of  $B$  from the density matrix of the full system in an mid-spectrum state. For a one dimensional lattice system of spinless fermions in the ergodic phase,  $S_2 \sim \frac{L}{2} - 1.2$  for system size  $L$  [21] and in the many-body localized phase,  $S_2 \sim L^0$ . Hence, the variation of  $(S_2 + 1.2)/L$  with  $h$  can be used as an efficient diagnostic to detect the thermal and MBL phases and the transition between them. This quantity is finite in the thermal phase and goes to zero in the MBL phase with increasing system size.

For model I with  $V = 1$  and  $\beta = -0.6$  and model II, with  $V = 1$  and  $\beta = -0.75$  as  $h$  is increased  $(S_2 + 1.2)/L$  decreases as shown in Fig.3. From the data the location of the thermal-MBL transition for the two models can be estimated and is consistent with their locations as obtained from energy-level spacing statistics. On the other hand, models III, IV and V do not display a thermal-MBL transition as shown in Fig. 4.

*Criterion for the occurrence of MBL:* Table. I summarizes the results of our calculations and the fifth column lists the values of  $\epsilon$  defined in Eqn. 1. The energy level spacing and entanglement entropy show that a

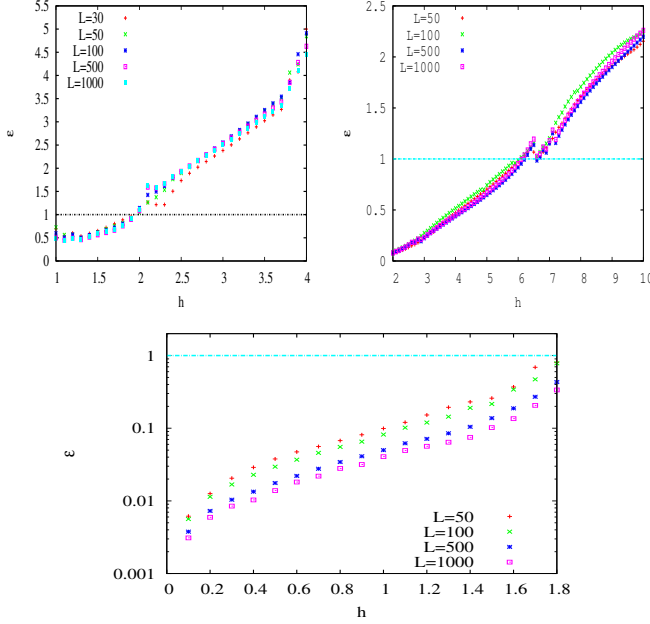


FIG. 5: (Color Online) The variation of  $\epsilon$  defined in Eqn. 1 with  $h$  as obtained for  $L = 50, 100, 500, 1000$  for (top left) model I ( $\beta = -0.6$ ), (top right) model II ( $\beta = -0.75$ ) without interactions ( $V = 0$ ). The dashed lines correspond to  $\epsilon = 1$ . The thermal-MBL transition for model I (II) has been estimated to be at  $h \approx 2(6)$  from the level spacing statistics and entanglement entropy of typical mid-spectrum states. It can be seen that  $\epsilon < 1$  for  $h$  below the transition value (thermal phase) and  $\epsilon > 1$  for  $h$  above the transition value (MBL phase). (Bottom)  $\epsilon$  as a function of  $h$  for model III ( $n = 0.5$ ) for  $V = 0$ . Model III always thermalizes upon introducing interactions. It can be seen that  $\epsilon < 1$  for this model for all values of  $h$ .

thermal-MBL transition occurs in models I and II but not models III, IV and V, which only have thermal phases. At the MBL transition for models I and II,  $\epsilon \sim 1$  as shown in Fig. 5 and in the MBL (thermal) phases  $\epsilon > (<)1$ . On the other for model III  $\epsilon < 1$  and this model always thermalizes. Similar values of  $\epsilon (< 1)$  are also obtained for models IV and V with  $V = 0$  which also always thermalize upon the introduction of interactions. Thus, our study shows that the quantity  $\epsilon$  can be used as a diagnostic to determine whether a system with a single particle mobility edge will display MBL upon the introduction of interactions. Further, the criterion for MBL to occur is  $\epsilon > 1$ .

*Conclusions and discussion:* We have investigated the effect of interactions on different models with mobility edges in the non-interacting limit using numerical exact diagonalization. We have demonstrated that MBL occurs in some of them (models I and II) but not in the

Model	MBL phase	Thermal phase	$\nu$	$\epsilon$
Model I	Yes	Yes	$< 1$	$> 1$ (in MBL phase) and $< 1$ (in thermal phase)
Model II	Yes	Yes	$< 1$	$> 1$ (in MBL phase) and $< 1$ (in thermal phase)
Model III	No	Yes	1	$< 1$
Model IV	No	Yes	$< 1$	$< 1$
Model V	No	Yes	$> 1$	$< 1$

TABLE I: A list of the models we study along which also shows whether they have thermal and MBL phases. The values of  $\nu$  and  $\epsilon$  for different models from exact diagonalization on systems of size  $L = 1000$ . The values of  $\nu$  agree with analytical results for the models for which they are available. The precise values of  $\nu$  can depend on specific parameters of the different models (see the Supplemental Material [36]) but they are always bounded in the way shown in the above table.

others (models III, IV and V) and have proposed a criterion for whether MBL occurs in a model with a single particle mobility edge upon the introduction of interactions. The relevant quantity to calculate is  $\epsilon$ , the weighted ratio of participation ratios of the delocalized and localized states as given in Eqn. 1 and the criterion is that MBL occurs when  $\epsilon > 1$  and the system thermalizes for  $\epsilon < 1$ . As mentioned earlier, it has been argued that the criterion for MBL to occur in a 1D system with a single particle mobility edge and a protected band of delocalized states upon introducing interactions is  $\nu \geq 1$  [29]. It is thus interesting to ask whether a similar criterion applies even to the models we study with no protected delocalized states. We have calculated  $\nu$  for these models for  $V = 0$  (listed in the fourth column of table I) using numerical exact diagonalization on systems up to  $L = 1000$  or known analytical results [26, 38, 39] (see the Supplemental Material [36]).

The actual value of  $\nu$  depends on the specific parameters of the model but is always bounded by or equal to 1 as indicated. It can be seen that both models I and II have  $\nu < 1$  but show a thermal to MBL transition as a function of  $h$  at a fixed value of the filling for fixed  $V$ . We have verified that such a transition holds down to values of  $V$  as low as 0.2 below which finite-size effects become pronounced (see the Supplemental Material [36]). Significantly,  $\nu$  is independent of  $h$  (which is the parameter that is tuned to effect a the thermal-MBL transition in models I and II) in the models we have studied. It thus follows that there is no critical value  $\nu_c$  such that models with  $\nu > \nu_c$  exist in one phase and those with  $\nu < \nu_c$  in



the other upon introducing weak interactions. We note however that all the models we study with  $\nu > 1$  do not display MBL. Finally, even though we have not been able to find a model with  $\nu > 1$  but  $\epsilon > 1$  to study, we predict that should such a model exist it will not display MBL upon the introduction of interactions.

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## APPENDIX A: LOCALIZATION LENGTH EXPONENT $\nu$

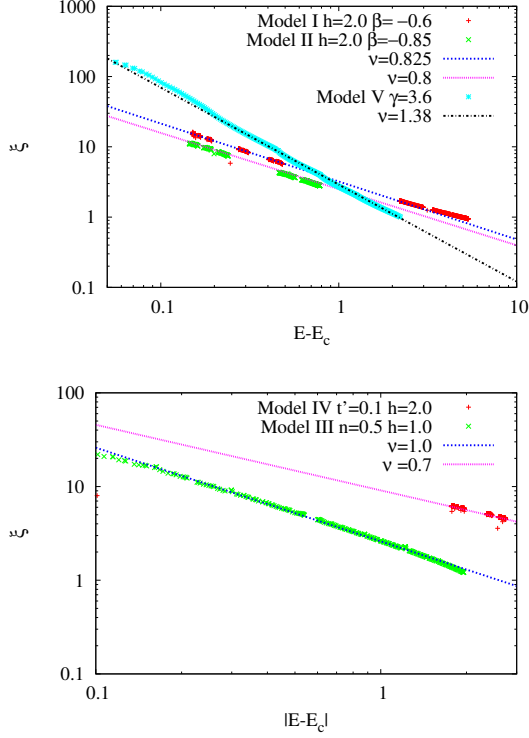


FIG. 6: Localization length  $\xi(E)$  of a state of energy  $E$  as function of energy  $E - E_c$  for models I, II, III, IV and V, where  $E_c$  is the mobility edge. The fits are to a form  $\xi(E) \sim |E - E_c|^\nu$ , with the value of  $\nu$  from the best fit indicated for each model. Note that the value of  $\nu$  for model IV cannot be obtained very reliably since the energy eigenvalues are not densely distributed over a sufficiently large range of  $|E - E_c|$ .

The localization length of an (exponentially) localized state is finite and whereas a delocalized state has infinite localization length in the thermodynamic limit. When bands of the two types of states are separated by a mobility edge at energy  $E_c$ , the localization length  $\xi(E)$  of a state with energy  $E$  on the localized side typically diverges as  $\xi(E) \sim 1/|E - E_c|^\nu$  close to the mobility edge. The values of  $\nu$  obtained numerically for models I, II, III, IV and V in the absence of interactions for system size  $L = 1000$  are shown in Fig. 6. The values obtained are  $\nu \approx 0.825$  (for  $\beta = -0.6$ ),  $\nu \approx 0.8$  (for  $\beta = -0.85$ ),  $\nu \approx 1$ ,  $\nu \approx 0.7$  and  $\nu \approx 1.38$  for models I, II, III, IV and V respectively. The values of  $\nu$  for model II, III and V have also been calculated earlier [26, 38, 39] and our calculations agree with those values. Note that the values of  $\nu$  for all models violate the Harris-Chayes criteria [40] which is  $\nu \geq 2/d$  where  $d$  is dimension of the system. This is not surprising as the bound only applies

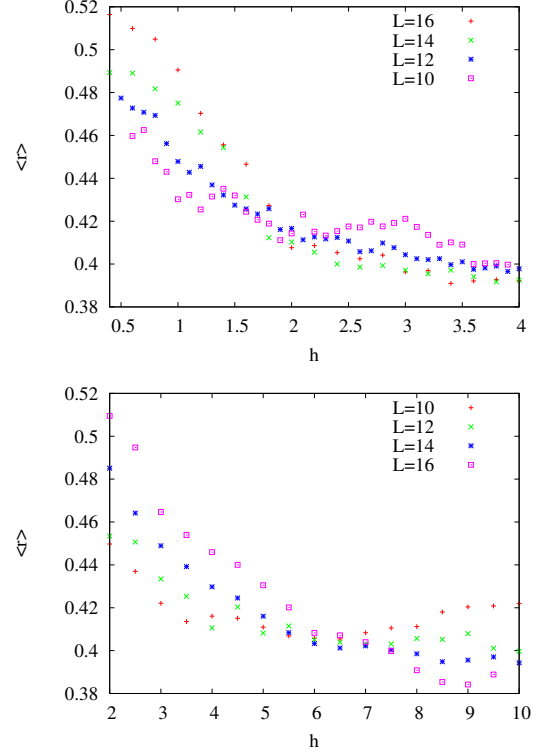


FIG. 7: (Color Online) The variation of the mean of the ratio between adjacent gaps in the spectrum with the strength of the incommensurate potential  $h$  for  $L = 10, 12, 14, 16$  at half filling for (top) model I with  $V = 0.2$  and  $\beta = -0.6$  and (bottom) model II with  $V = 0.2$  and  $\beta = -0.75$ .

to systems with uncorrelated disorder which none of our models have.

## APPENDIX B: RESULTS FOR $V=0.2$

We have shown in the main manuscript that models I and II have a thermal-MBL transition as a function of disorder strength  $h$  at fixed interaction strength  $V = 1.0$ . The transitions occur at  $h \approx 2$  and  $h \approx 6$  for model I and model II respectively. We use this result along with the fact that  $\epsilon = 1$  at the same values of  $h$  for the two models to argue that the single particle spectrum can indicate whether thermalization or many-body localization will occur upon the introduction of *weak* interactions. Here we strengthen that claim by showing the transitions also occur at the much smaller value  $V = 0.2$  for the models (as can be seen in Fig. 7) at approximately the same values of  $h$  as for  $V = 1.0$ . The diagnostic used is the energy level spacing distribution and the particular quantity that has been calculated to locate the transition is the mean value of the ratio between adjacent gaps, as explained in the main manuscript. However, we

note that finite-size effects are more pronounced at this smaller value of  $V$ , especially on the thermal side (i.e. small values of  $h$ ) than for  $V = 1.0$ . Such finite-size

effects have been investigated previously in the context of crossovers between different level spacing distributions upon the introduction of a perturbation [41, 42].